

LARGE DEVIATIONS FOR THE PERIMETER OF CONVEX HULLS OF PLANAR RANDOM WALKS

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ABSTRACT. We give logarithmic asymptotic bounds for large deviations probabilities for perimeter of the convex hull of a planar random walk. These bounds are sharp for a wide class of distributions of increments that includes Gaussian distributions and shifted or linearly transformed rotationally invariant distributions. For such random walks, large deviations of the perimeter are attained by the trajectories that asymptotically align into line segments. These results on the perimeter are easily extended to mean width of convex hulls of random walks in higher dimensions. Our method also allows to find the logarithmic asymptotics of large deviations probabilities for area of the convex hull of planar random walks with rotationally invariant distributions of increments.

1. INTRODUCTION AND MAIN RESULTS

1.1. Motivation. Let $S_k = X_1 + \dots + X_k$ be a planar random walk with independent identically distributed increments X_1, X_2, \dots . We assume that expectation of X_1 exists and is finite, and put $\mu := \mathbb{E}X_1$. Denote by P_n perimeter of the convex hull $C_n := \text{conv}(0, S_1, \dots, S_n)$ of the first n steps of the random walk including the origin; by perimeter of a line segment we mean its doubled length. It is intentional that we do not exclude one-dimensional distributions (i.e. those concentrated on affine lines of the plane) from our consideration as we will use them to construct examples. Although all of our results remain valid for the convex hulls $\text{conv}(S_1, \dots, S_n)$, it is traditional to consider C_n .

Let us briefly describe the known results on perimeter of C_n . The remarkable Spitzer–Widom formula [13] states that

$$\mathbb{E}P_n = 2 \sum_{k=1}^n \frac{\mathbb{E}|S_k|}{k}, \quad (1)$$

where by $|\cdot|$ we denote the Euclidean norm. This implies that $\mathbb{E}P_n/n \rightarrow 2|\mu|$, as follows by the law of large numbers for S_k and the uniform integrability of S_k/k . Wade and Xu [16] developed the ideas introduced by Snyder and Steele [12] and showed that if $\mu \neq 0$ and $\mathbb{E}|X_1|^2 < \infty$, then $\text{Var}(P_n)/n \rightarrow 4 \frac{\mathbb{E}(\mu \cdot X_1)^2 - |\mu|^4}{|\mu|^2}$ so the variance of the perimeter grows linearly. Moreover, if $\mathbb{E}|X_1|^2 < \infty$, then P_n satisfies a central limit theorem – this result is proved

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in [16] for $\mu \neq 0$ and in [17] for $\mu = 0$; in the later case the CLT naturally follows from the invariance principle.

The Spritzer–Widom formula (1) admits various generalizations to higher dimensions, including the formulas for expected mean width, surface area, volume, and other intrinsic volumes of the multidimensional convex hull, see Barndorff-Nielsen and Baxter [2] and Vysotsky and Zaporozhets [15].

In this paper we are interested in large deviations probabilities $\mathbb{P}(P_n \geq 2xn)$ for $x > |\mu|$ and $\mathbb{P}(P_n \leq 2xn)$ for $x < |\mu|$. We are also interested in the shape of the trajectories resulting in such large deviations of the perimeter. To the best of our knowledge, there is only one rigorous result on large deviations of the perimeter – it is due to Snyder and Steele [12], who obtained a non-sharp upper bound for random walks with bounded increments: if $|X_1| \leq M$ a.s., then

$$\mathbb{P}(|P_n - \mathbb{E}P_n| \geq xn) \leq 2 \exp(-x^2 n / (8\pi^2 M^2)), \quad x \geq 0.$$

The recent paper of Claussen et al. [3] (followed by the work of Dewenter et al. [6] that concerns joint convex hulls of several random walks) does some numerical analysis of atypically large values of the perimeter and concludes that the perimeter “seems ... to obey the large-deviation principle” for walks with Gaussian increments. We prove this conclusion for a wide class of random walks, which includes Gaussian walks, and provide estimates for large deviations probabilities that are valid for general walks with the finite Laplace transform of their increments.

1.2. The notation. Recall that the *Legendre–Fenchel transform* or the *convex conjugate* of a function $F : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ (where $d \geq 1$) with the non-empty effective domain $\mathcal{D}_F := \{u : F(u) < \infty\}$ is the function $F^* : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$F^*(v) := \sup_{u \in \mathbb{R}^d} (u \cdot v - F(u)), \quad v \in \mathbb{R}^d.$$

The conjugate F^* is convex and lower semi-continuous on \mathbb{R}^d (F itself does not need to be convex). Recall that any convex function F is continuous on the relative interior $\text{rint } \mathcal{D}_F$ of its effective domain (Rockafellar [10, Theorem 10.1]) so the property of lower semi-continuity characterizes F near the relative boundary of \mathcal{D}_F . By $\text{conv } F$ we denote the *largest convex minorant* or the *convex hull* of the function F , i.e. the convex function with the epigraph $\text{conv}(\text{epi } F)$ in \mathbb{R}^{d+1} .

The main assumption of this paper is that the Laplace transform $\mathcal{L}(u) := \mathbb{E}e^{u \cdot X_1}$ is finite for all $u \in \mathbb{R}^2$. The cumulant generating function $K(u) := \log \mathbb{E}e^{u \cdot X_1}$ is then convex, infinitely differentiable, and satisfies $K(0) = 0$. Its convex conjugate $I := K^*$ is the *rate function* of X_1 . It is non-negative, lower semi-continuous, satisfies $I(\mu) = 0$, and is continuous on $\text{rint } \mathcal{D}_I$.

Define the *radial maximum* and *radial minimum* functions $\bar{(\cdot)}$ and $\underline{(\cdot)}$ as

$$\underline{I}(r) := \inf_{\ell: |\ell|=1} I(r\ell), \quad \bar{\mathcal{L}}(p) := \max_{\ell: |\ell|=1} \mathcal{L}(p\ell), \quad p, r \geq 0,$$

and put $\underline{I}(r) = \bar{\mathcal{L}}(p) := \infty$ for $p, r < 0$. Clearly, the maximum and the infimum are always attained at some points since the Laplace transform is continuous, I is lower semi-continuous, and the circles $r\mathbb{S}^1, p\mathbb{S}^1$ are compact. Thus the respective sets of minimal and

maximal directions

$$\underline{\Lambda}_r(I) := \operatorname{argmin}_{\ell: |\ell|=1} I(r\ell), \quad \bar{\Lambda}_p(\mathcal{L}) := \operatorname{argmax}_{\ell: |\ell|=1} \mathcal{L}(p\ell), \quad r, p > 0$$

are always non-empty. Without any risk of confusion, throughout this section we will use the short notation $\underline{\Lambda}_r$ and $\bar{\Lambda}_p$ for $\underline{\Lambda}_r(I)$ and $\bar{\Lambda}_p(\mathcal{L}) = \bar{\Lambda}_p(K)$, respectively.

Since the radial minimum function \underline{I} and the sets of minimal directions $\underline{\Lambda}_r$ appear in our main result, it is convenient to state some of their basic properties. Denote by $\operatorname{supp}(X)$ the topological support of the distribution of a random vector X , that is the smallest closed set B such that $\mathbb{P}(X \in B) = 1$. Define

$$r_{\min} := \min\{|x| : x \in \operatorname{conv}(\operatorname{supp}(X_1))\}, \quad r_{\max} := \max\{|x| : x \in \operatorname{supp}(X_1)\}.$$

Let us agree that by $[[\mu], r_{\max}]$ we will mean the half-line $[[\mu], \infty)$ if $r_{\max} = \infty$.

Lemma 1. *Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$. Then*

- a) *The effective domain $\mathcal{D}_{\underline{I}}$ of \underline{I} is an interval that satisfies $\operatorname{int} \mathcal{D}_{\underline{I}} = (r_{\min}, r_{\max})$;*
- b) *The function \underline{I} is lower semi-continuous, satisfies $\underline{I}(|\mu|) = 0$, is strictly decreasing and convex on $[r_{\min}, |\mu|]$, and is strictly increasing on $[[\mu], r_{\max}]$;*
- c) *For any $r \in (r_{\min}, |\mu|]$, the set $\underline{\Lambda}_r$ contains a unique element, which we denote by ℓ_r .*

Note that \underline{I} admits the following geometric interpretation: the epigraph of $\underline{I}(|v|)$ is the union of all rotations of the epigraph of $I(v)$ about the vertical axis.

Example 1. The function \underline{I} is not necessarily continuous on $[[\mu], r_{\max})$: it is easy to check that if $\mathbb{P}(X_1 = (1, 0)) = 3/4$ and $\mathbb{P}(X_1 = (-2, 0)) = 1/4$, then \underline{I} has a jump at $r = 1$. It is also possible to show that \underline{I} is discontinuous for the “truly” two-dimensional distribution that is a mixture of the above two-atomic distribution and the uniform distribution on the disk $\{u : |u| \leq 1\}$. We will give a sufficient condition for the continuity of \underline{I} in Lemma 2 of the next section.

In these examples of course $\mu \neq 0$. It is not entirely clear if \underline{I} can be discontinuous for centred distributions but it certainly can be non-convex, see Example 2 in Section 2.4 below.

1.3. Main results. We are now ready to state the main result.

Theorem 1. *Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$.*

- 1. *For any $x \in (r_{\min}, |\mu|]$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(P_n \leq 2xn) = -\underline{I}(x), \quad (2)$$

and moreover, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \frac{S_k}{n} - \frac{k}{n} x \ell_x \right| \leq \varepsilon \mid P_n \leq 2xn\right) = 1. \quad (3)$$

- 2. *For any $x \in [[\mu], r_{\max})$, we have*

$$-\underline{I}(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(P_n \geq 2xn) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(P_n \geq 2xn) \leq -\operatorname{conv} \underline{I}(x). \quad (4)$$

If in addition $\underline{I}(x) = \text{conv } \underline{I}(x)$, then for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{1 \leq k \leq n} \left| \frac{S_k}{n} - \frac{k}{n} x \ell \right| \leq \varepsilon \text{ for some } \ell \in \underline{\Lambda}_x \mid P_n \geq 2xn \right) = 1. \quad (5)$$

Remark 1. Consider the boundary cases $x = r_{\min}$ and $x = r_{\max}$.

Since $P_n \geq 2r_{\min}n$ a.s., it clearly holds

$$\mathbb{P}(P_n \leq 2r_{\min}n) = \mathbb{P}(S_k = ku_0, k = 1, \dots, n) = (\mathbb{P}(X_1 = u_0))^n,$$

where u_0 is the closest point of $\text{conv}(\text{supp}(X_1))$ to the origin. Note that it is possible that $\underline{I}(r_{\min}) = I(u_0) \neq \log \mathbb{P}(X_1 = u_0)$ so (2) may not hold at $x = r_{\min}$. However, it does hold if either $\mathbb{P}(X_1 = u_0) = 0$ or $\mathbb{P}(X_1 = u_0) > 0$ and u_0 is an extremal point of $\text{conv}(\text{supp}(X_1))$, see Lemma 2.a of Section 2.

We will also see in Lemma 2.b that if $0 < r_{\max} < \infty$, then

$$\underline{I}(r_{\max}) = -\log \left(\max_{\ell: |\ell|=1} \mathbb{P}(X_1 = r_{\max}\ell) \right), \quad \underline{\Lambda}_{r_{\max}} = \operatorname{argmax}_{\ell: |\ell|=1} \mathbb{P}(X_1 = r_{\max}\ell).$$

In that case by $P_n \leq 2r_{\max}n$ a.s., we have

$$\mathbb{P}(P_n \geq 2r_{\max}n) = \mathbb{P}(S_k = kr_{\max}\ell, k = 1, \dots, n \text{ for some } \ell \in \underline{\Lambda}_{r_{\max}}) = \#(\underline{\Lambda}_{r_{\max}})e^{-n\underline{I}(r_{\max})},$$

where by the definition the last expression equals 0 if $\underline{I}(r_{\max}) = \infty$.

Note that Theorem 1 gives the logarithmic asymptotics of the large deviation probabilities for $x \leq |\mu|$ without any additional assumptions on the rate function I but for $x \geq |\mu|$, we assumed that $\underline{I}(x) = \text{conv } \underline{I}(x)$. This condition means that the epigraph of \underline{I} admits a support line at the point $(x, \underline{I}(x))$. This is true for every x iff \underline{I} is convex, which is not always the case as commented above. However, we think that this convexity assumption is not really needed in the following sense.

Conjecture 1. Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$. For any $x \in [|\mu|, r_{\max})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(P_n \geq 2xn) = -\underline{I}(x)$$

and (5) holds true.

Here is an informal explanation of why this is true for distributions with convex \underline{I} . In this case, the standard large deviations approach reduces the problem to finding a curve of minimal length with fixed perimeter of its convex hull. It is natural that this minimum is attained on line segments.

We now present few types of distributions that satisfy the conclusion of Conjecture 1. Let $\Sigma := \mathbb{E}(X_1 X_1^\top) - \mu \mu^\top$ denote the covariance matrix of X_1 . Recall that the support of the distribution of X_1 has dimension $\text{rank } \Sigma$.

Proposition 1. The function \underline{I} is convex in either of the following cases:

- a) Σ is non-degenerate and $\Sigma^{-1/2}X_1$ has a rotationally invariant distribution;
- b) μ is a non-zero maximal eigenvector of Σ and $\Sigma^{-1/2}(X_1 - \mu)$ has a rotationally invariant distribution;

c) X_1 is Gaussian(μ, Σ).

Therefore Conjecture 1 is valid the cases above and, in addition, in the case that

d) X_1 is supported on an affine line.

We were able to prove the convexity of \underline{I} only for *affine* transforms of a standard Gaussian distribution but, unfortunately, not of a general rotationally invariant distribution. By affine transforms we mean compositions of linear transforms and translations. Let us repeat that $\underline{I}(x)$ is not necessarily convex in Case d.

The following result gives a simple description of $\text{conv } \underline{I}$ directly in terms of \mathcal{L} , the Laplace transform of increments. In particular, we will use it to prove the above stated convexity of \underline{I} for Gaussian distributions.

By $\text{im}(\cdot)$ we will denote the image of a function, that is the set of its possible values as the argument varies over the effective domain. By $(\cdot)'_+$ and $(\cdot)'_-$ we denote the right and the left derivatives of a function of real argument.

Proposition 2. *Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$. Then*

a) $\bar{K} = \log \bar{\mathcal{L}}$ is an increasing convex function on $[0, \infty)$;

b) For any $p > 0$, the one-sided derivatives satisfy

$$\bar{K}'_+(p) = \max_{\ell \in \bar{\Lambda}_p} |\nabla K(p\ell)| \quad \text{and} \quad \bar{K}'_-(p) = \min_{\ell \in \bar{\Lambda}_p} |\nabla K(p\ell)|.$$

Importantly, for any $r \geq |\mu|$,

c) $\text{conv } \underline{I}(r) = (\bar{K})^*(r)$;

d) If $r \in \text{cl}(\text{im}(\bar{K}'))$, then $\underline{I}(r) = \text{conv } \underline{I}(r) < \infty$;

e) If $r \in \text{im}(\bar{K}')$ and $p > 0$ is such that $\bar{K}'(p) = r$, then $\underline{\Lambda}_r = \bar{\Lambda}_p$.

Corollary 1. \underline{I} is convex if and only if \bar{K} (or $\bar{\mathcal{L}}$) is differentiable on $(0, \infty)$.

Corollary 2. \bar{K} is differentiable if there exists a continuous mapping $\ell : (0, \infty) \rightarrow \mathbb{S}^1$ such that $\ell(p) \in \bar{\Lambda}_p$ for any $p > 0$.

The main result here is Part c, which relates the radial maximum of the Laplace transform to the radial minimum of its convex conjugate. This is actually a general fact valid for any convex function, see Proposition 3 of the next section. Although it is quite possible that this result is already known in convex analysis, we present it here as we did not find a reference. Part d is an easily consequence of Part c and the well-known fact that the Legendre–Fenchel transform maps kinks of a convex function (in our case, \bar{K}) into linear segments of its convex conjugate. Part e, which is another natural consequence of Part c, claims that the slowest directions of I are exactly the fastest directions of \mathcal{L} at the corresponding radii.

The condition of Corollary 2 is satisfied if the set $\cap_{p>0} \bar{\Lambda}_p$ is non-empty, i.e. there exists a direction that maximizes the Laplace transform at all radii. This restrictive assumption naturally holds true for linearly transformed or shifted rotationally invariant distributions of increments, i.e. those described by respective Cases a and b of Proposition 1. We will see that for distributions of the first type, each $\bar{\Lambda}_p$ coincides with the set of unit maximal eigenvectors of the covariance matrix Σ , and for the second type, it holds $\bar{\Lambda}_p = \{\mu/|\mu|\}$ for any $p > 0$. We will also show that a non-degenerate Gaussian distribution satisfies the

assumption of Corollary 2 but the set $\cap_{p>0} \bar{\Lambda}_p$ is empty unless the distribution is one of the two types described above.

1.4. Further extensions. Let A_n denote area of the convex hull C_n . There is a simple expression for $\mathbb{E}A_n$ similar to the Spitzer–Widom formula (1), see Barndorff-Nielsen and Baxter [2] or Vysotsky and Zaporozhets [15]. Further, the invariance principle naturally implies that A_n satisfy a limit theorem (under the scaling n^{-1} for $\mu = 0$ and $n^{-3/2}$ for $\mu \neq 0$), as proved by Wade and Xu [17]. Numerical studies of large deviations probabilities of A_n were performed by Claussen et al. [3], who arrived to the same conclusion as for the perimeter, namely that the area follows a large deviations principle.

The main idea used in our proof of Theorem 1 can be applied to find large deviations probabilities of A_n for random walks with rotationally invariant distributions of increments. The question of finding a planar curve of unit length that maximizes the area of its convex hull is known as Ulam’s problem. Although it is very similar to the classical Dido problem and of course has the same answer that the curve is a half-circle as proved directly by Moran [9], we think that there is no easy solution by reduction to the Dido problem. The corresponding isoperimetric inequality easily yields the following result.

Theorem 2. *Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$. Suppose that X_1 has a rotationally invariant distribution. Then for any $a \in [0, r_{max}^2/(2\pi))$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(A_n \geq an^2) = -I(\sqrt{2\pi a}), \quad (6)$$

and moreover, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \frac{S_k}{n} - \sqrt{\frac{2a}{\pi}} (\cos(\sigma\pi k/n + \alpha) - \cos \alpha, \sin(\sigma\pi k/n + \alpha) - \sin \alpha) \right| \leq \varepsilon \right. \\ \left. \text{for some } \sigma \in \{-1, +1\} \text{ and } \alpha \in \mathbb{R} \mid A_n \geq an^2 \right) = 1. \quad (7)$$

Thus large deviations of the area are attained on the trajectories that asymptotically align into half-circles and move with a constant speed.

It seems that if the distribution of $X_1 - \mu$ is rotationally invariant and $\mu \neq 0$, then the limit shapes should not be universal, unlike the case of the perimeter. It is very plausible that for the particular case of shifted standard Gaussian increments, these limit shapes still are circular arcs. We present an incomplete argument in Example 3 of Section 3.

Remark 2. The theorem remains valid for X_1 such that its covariance matrix Σ is non-degenerate and $\Sigma^{-1/2}X_1$ has a rotationally invariant distribution if we substitute r_{max}^2 , $\sqrt{2\pi a}$, and $\sqrt{\frac{2a}{\pi}}$ by area of $\text{supp}(X_1)$, $\sqrt{2\pi a(\det \Sigma)^{-1/2}}$, and $\sqrt{\frac{2a}{\pi(\det \Sigma)^{1/2}}} \Sigma^{1/2}$, respectively. Then the limit shapes of the trajectories are halves of the ellipse $\text{supp}(X_1)$ divided by lines passing through its centre.

This statement follows from the fact that $\Sigma^{-1/2}S_k, k \geq 1$, is a random walk with rotationally invariant increments and the scaling

$$\text{area}(\text{conv}(\Sigma^{-1/2}S_1, \dots, \Sigma^{-1/2}S_n)) = (\det \Sigma)^{-1/2} A_n.$$

It makes sense to consider large deviations of surface area, volume, etc. for convex hulls of random walks in higher dimensions. The expected values of these characteristics are available through the explicit formulas of [15] that generalize (1). However, currently we can not obtain any progress even for centred rotationally invariant distributions of increments. In fact, according to Tilli [14], the problem of finding the shape of a curve in \mathbb{R}^3 of unit length that maximizes volume of its convex hull is solved only in the class of convex curves (those that do not have four coplanar points) and there is no complete solution. Croft et al. [4, Problem A28] mention that there are no results on the similar problem of maximizing the surface area, and we are unaware of any progress in this direction.

Remark 3. On the other hand, our results for the perimeter of planar random walks can be easily extended for *mean width* of convex hulls in higher dimensions. A closely related property of a convex set is its *first intrinsic volume*, which equals (see [15, Eq. (23)]) the mean width divided by $\frac{2v_{d-1}}{dv_d}$, which is mean width of a unit segment, where v_d denotes volume of a unit ball in \mathbb{R}^d , $d \geq 2$.

Assume now that S_n is a random walk in \mathbb{R}^d and extend in the obvious way the definition of \underline{I} and the other quantities related to the Laplace transform $\mathbb{E}e^{u \cdot X_1}$. Lemma 1 of course holds in this setting.

Denote by W_n and V_n mean width and first intrinsic volume, respectively, of the convex hull C_n . The Spitzer–Widom formula (1) remains valid (see [15]) in any dimension if we replace perimeter P_n of the convex hull C_n by its doubled first intrinsic volume $2V_n$. Accordingly, our Theorem 1 remains valid if we replace P_n by $\frac{dv_d}{v_{d-1}}W_n = 2V_1$ so the probabilities change to $\mathbb{P}(W_n \geq \frac{2v_{d-1}}{dv_d}xn)$ and $\mathbb{P}(W_n \leq \frac{2v_{d-1}}{dv_d}xn)$ or, equivalently, more elegant $\mathbb{P}(V_n \geq xn)$ and $\mathbb{P}(V_n \leq xn)$. Clearly, Conjecture 1 can be reformulated for W_n , and it is valid in the cases described by the d -dimensional version of Proposition 1, which also holds true.

2. CONVEX-ANALYTIC CONSIDERATIONS

2.1. Basic facts from convex analysis. Suppose that $F : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is any function with a non-empty effective domain \mathcal{D}_F . By [10, Theorem 12.2]) it holds

$$F^{**} = \text{cl}(\text{conv } F), \quad (8)$$

where $\text{cl}(\cdot)$ denotes the *closure* of a function, i.e. the function with the epigraph $\text{cl}(\text{epi}(\cdot))$. Recall that F is lower semi-continuous iff $F = \text{cl } F$, i.e. its epigraph $\text{epi } F$ is closed in \mathbb{R}^{d+1} ([10, Theorem 7.1]). Thus the Legendre–Fenchel transform is an involution on the set of convex lower semi-continuous functions.

The *Fenchel inequality* $F(u) + F^*(v) \geq u \cdot v$, which holds for any $u, v \in \mathbb{R}^d$, immediately follows from the definition of convex conjugation.

Suppose that $d = 1$ and the function F is convex and finite on \mathbb{R} . We claim that F^* is affine on any interval contained in the set $\text{conv}(\text{cl}(\text{im}(F')) \setminus \text{cl}(\text{im}(F')))$ and there are no other intervals of affinity of F^* . Thus kinks of convex functions correspond to affine segments of their convex conjugates, and vice versa. The convex hull appears here due to $\mathcal{D}_{F^*} = \text{conv}(\text{cl}(\text{im}(F')))$.

In fact, if F is not differentiable at a $u \in \mathbb{R}$, then

$$\inf \text{im}(F') \leq F'_-(u) \neq F'_+(u) \leq \sup \text{im}(F'). \quad (9)$$

It is easy to see that F^* is affine on $[F'_-(u), F'_+(u)]$ with slope u , and the first claim follows since both $F'_-(u)$ and $F'_+(u)$ belong to $\text{cl}(\text{im}(F'))$. To check the second statement, assume that F^* is affine on some non-empty interval with some slope u . By taking the Legendre–Fenchel transform, we obtain that for F^{**} , which equals F , the relations $F'_+(u) - F'_-(u) > 0$ and (9) hold true.

Finally, if $d \geq 1$ and the function F is convex, finite, and differentiable on \mathbb{R}^d , then F is continuously differentiable (see [10, Corollary 25.5.1]) and F^* is strictly convex on $\text{rint}(\mathcal{D}_{F^*})$, i.e. the graph of F^* contains no line segments ([10, Theorem 26.3]). Note that it follows from Lemma 2.a that the rate function I is actually strictly convex on \mathcal{D}_I (if $\mathcal{D}_{\mathcal{L}} = \mathbb{R}^d$), but we will not use this statement.

2.2. Basic properties of the radial minimum function \underline{I} . We prove the d -dimensional version of Lemma 1.

Proof of Lemma 1. a) Since \mathcal{D}_I is a convex set satisfying $\text{cl } \mathcal{D}_I = \text{conv}(\text{supp}(X_1))$ (this is explained below in (10)), it readily follows that $\text{cl } \mathcal{D}_I = (r_{\min}, r_{\max})$ and of course \mathcal{D}_I is convex.

b) Clearly, $\underline{I}(|\mu|) = 0$ follows by $I(\mu) = 0$. Since I is strictly convex on $\text{rint } \mathcal{D}_I$ and attains its minimum at μ , we conclude that \underline{I} strictly decreases on $[r_{\min}, |\mu|]$ and strictly increases on $[|\mu|, r_{\max}]$. This is because the line segment that joins a point of the sphere $r\mathbb{S}^{d-1}$ with μ always intersects the sphere $r'\mathbb{S}^{d-1}$ if $0 \leq r < r' < |\mu|$ or $|\mu| < r' < r$.

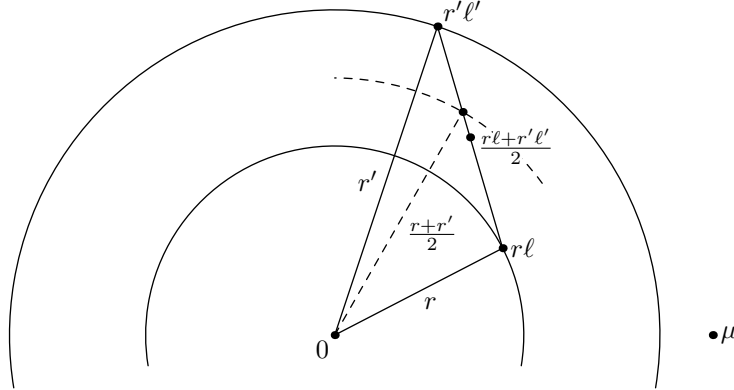


Fig. 1.

The function \underline{I} is convex on $[r_{\min}, |\mu|]$ since for any $r_{\min} \leq r < r' \leq |\mu|$ and $\ell \in \underline{\Delta}_r, \ell' \in \underline{\Delta}_{r'}$, it holds

$$\underline{I}(r) + \underline{I}(r') = I(r\ell) + I(r'\ell') \geq 2I\left(\frac{r\ell + r'\ell'}{2}\right) \geq 2\underline{I}\left(\left|\frac{r\ell + r'\ell'}{2}\right|\right) \geq 2\underline{I}\left(\frac{r + r'}{2}\right).$$

Here we used the triangle inequality and the fact that \underline{I} decreases on $[0, |\mu|]$, see Figure 1 for a geometric explanation in the planar case.

It is easy to see that \underline{I} is right-continuous on $[r_{\min}, |\mu|]$ (actually, \underline{I} is continuous on $(r_{\min}, |\mu|]$ since it is convex on this interval) and left-continuous on $[|\mu|, r_{\max}]$. This follows from the fact that \underline{I} is monotone on these intervals combined with the lower semi-continuity of the rate function I and a standard compactness argument. Then the left- and right-continuity and the monotonicity properties imply that \underline{I} is lower semi-continuous.

c) Suppose that for an $r \in (r_{\min}, |\mu|]$, there are two distinct elements ℓ, ℓ' in $\underline{\Lambda}_r$. By the convexity of I , it holds $\underline{I}(r|\ell + \ell'|/2) \leq I(r(\ell + \ell')/2) \leq \underline{I}(r)$, which is a contradiction by Part b since \underline{I} is strictly decreasing on $(r_{\min}, |\mu|]$ and $|\ell + \ell'| < 2$. \square

2.3. Effective domains of rate functions and the continuity of \underline{I} . The effective domain \mathcal{D}_I of the rate function I is a convex set that is known to satisfy

$$\text{cl } \mathcal{D}_I = \text{conv}(\text{supp}(X_1)). \quad (10)$$

With a certain effort, this follows from (20) of the next section. So $\text{conv}(\text{supp}(X_1))$ differs from \mathcal{D}_I by a subset of its relative boundary. This subset is described in the first part of the following result.

In the second part we give a sufficient condition for the continuity of \underline{I} . Recall that by Lemma 1.b, this function is continuous on $[r_{\min}, |\mu|]$ but only left-continuous on $[|\mu|, r_{\max}]$.

Lemma 2. *Assume that $\mathcal{L}(u) = \mathbb{E}e^{u \cdot X_1} < \infty$ for any $u \in \mathbb{R}^2$.*

- a) *Suppose that $v \in \mathcal{D}_I \setminus \text{rint } \mathcal{D}_I$. Then for any support line l to $\text{conv}(\text{supp}(X_1))$ at v it is true that $\mathbb{P}(X_1 \in l) > 0$ and $I(v) = (\log \mathbb{E}[e^{u \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}])^*(v)$. Moreover, if v is an extremal point of $\text{conv}(\text{supp}(X_1))$, then $\mathbb{P}(X_1 = v) > 0$ and $I(v) = -\log \mathbb{P}(X_1 = v)$.*
- b) *Suppose that \underline{I} is discontinuous at an $x \in [|\mu|, r_{\max}]$. Then for any $\ell \in \underline{\Lambda}_x$, $x\ell$ is an extremal point of $\text{conv}(\text{supp}(X_1))$ and $\underline{I}(x) = -\log \mathbb{P}(X_1 = x\ell) < \infty$.*

Part a means that on the line l , the rate function I coincides with the rate function of the non-zero sub-probability distribution $\mathbb{P}(X_1 \in \cdot \cap l)$. This new rate function is of course infinite on $\mathbb{R}^2 \setminus l$. Part b generalizes Example 1 of the previous section.

Proof. a) Let l be a support line to $\text{conv}(\text{supp}(X_1))$ at v . Take ℓ to be either of the two directions parallel to l , and let ℓ^\perp be the direction orthogonal to l such that $\ell^\perp \cdot u \leq \ell^\perp \cdot v$ for any $u \in \text{conv}(\text{supp}(X_1))$, see Figure 2. Put $v_1 := \ell \cdot v$ and $v_2 := \ell^\perp \cdot v$ so $v = (v_1, v_2)$ in the orthonormal basis ℓ, ℓ^\perp .

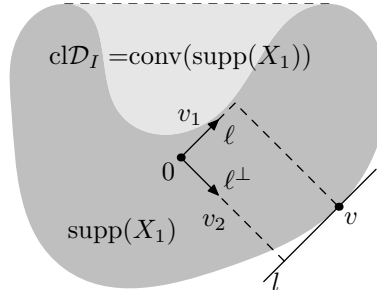


Fig. 2.

Let us fix a $u_1 \in \mathbb{R}$. Note that the function $u_2 \mapsto \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}$ is convex on \mathbb{R} as the restriction of the convex function $K(u)$ defined on \mathbb{R}^2 to the line $u_1 \ell + \text{Lin}(\ell^\perp)$. Since $\ell^\perp \cdot X_1 \leq v_2$ a.s., it holds

$$\frac{\partial}{\partial u_2} \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1} = \frac{\mathbb{E}[(\ell^\perp \cdot X_1) e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}]}{\mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}} \leq v_2$$

and the last inequality is strict unless $\mathbb{P}(X_1 \in l) = 1$. In the later case the function $u_2 \mapsto u_2 v_2 - \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}$ is constant, otherwise it is concave and strictly increasing on \mathbb{R} . Thus it always holds

$$\begin{aligned} \sup_{u_2 \in \mathbb{R}} (u_2 v_2 - \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}) &= \lim_{u_2 \rightarrow \infty} (u_2 v_2 - \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}) \\ &= -\log \left(\lim_{u_2 \rightarrow \infty} \mathbb{E} e^{u_1 \ell \cdot X_1 + u_2 (\ell^\perp \cdot X_1 - v_2)} \right) = -\log \mathbb{E} [e^{u_1 \ell \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}], \end{aligned}$$

with the last equation following by the monotone convergence theorem. This implies that

$$I(v) = \sup_{u_1 \in \mathbb{R}} \sup_{u_2 \in \mathbb{R}} (u_1 v_1 + u_2 v_2 - \log \mathbb{E} e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1}) = \sup_{u_1 \in \mathbb{R}} (u_1 v_1 - \log \mathbb{E} [e^{u_1 \ell \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}]),$$

where the last expression is exactly the value of the Legendre–Fenchel transform of

$$F(u) = \log \mathbb{E} [e^{u \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}] = \log \mathbb{E} [e^{(u_1 \ell + u_2 \ell^\perp) \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}] = \log \mathbb{E} [e^{u_1 \ell \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}] + u_2 v_2$$

at the point $v = v_1 \ell + v_2 \ell^\perp$. It remains to note that the expectation under the last supremum is strictly positive for all u_1 and thus $\mathbb{P}(X_1 \in l) > 0$ because otherwise it would be that $I(v) = \infty$ contradicting the assumption $v \in \mathcal{D}_I$.

Now assume that v is an extremal point of $\text{conv}(\text{supp}(X_1))$. Then $l \cap \text{conv}(\text{supp}(X_1))$ lies to one side of v on l , and we can assume without loss of generality that $\ell \cdot u \leq \ell \cdot v = v_1$ for any $u \in l \cap \text{conv}(\text{supp}(X_1))$. The rest is exactly as above: the function $u_1 \mapsto u_1 v_1 - \log \mathbb{E} [e^{u_1 \ell \cdot X_1} \mathbb{1}_{\{X_1 \in l\}}]$ is concave and strictly increasing unless $\mathbb{P}(X_1 \in l) = \mathbb{P}(X_1 = v)$, in which case it is constant.

b) The case $X_1 = \mu$ a.s. is trivial and we exclude it from the following consideration. Now $\text{rint } \mathcal{D}_I$ is non-empty and contains an open interval that includes μ . Then \underline{I} is continuous at $|\mu|$ by the continuity of I on $\text{rint } \mathcal{D}_I$. Further, if $x = r_{\max}$, then necessarily $\underline{I}(x) < \infty$, otherwise \underline{I} is continuous at r_{\max} by the lower semi-continuity of I and a standard compactness argument.

Thus $\underline{I}(x) < \infty$ and for any $\ell \in \underline{\Lambda}_x$, it holds $x\ell \in \mathcal{D}_I \setminus \text{rint } \mathcal{D}_I$. Indeed, if $x\ell \in \text{rint } \mathcal{D}_I$, then \underline{I} is continuous at x since I is continuous on $\text{rint } \mathcal{D}_I$, which contains an open interval that includes $x\ell$. If $x\ell$ is not an extremal point of \mathcal{D}_I , then for the unique support line l of $\text{conv}(\text{supp}(X_1))$ at $x\ell$ it holds $x\ell \in \text{rint}(l \cap \text{conv}(\text{supp}(X_1)))$. Since by Part a, $I|_l$ is continuous on $\text{rint}(l \cap \text{conv}(\text{supp}(X_1)))$ as the rate function of a sub-probability distribution, \underline{I} must be continuous at x and we arrive to a contradiction. Thus $x\ell$ is an extremal point of $\text{conv}(\text{supp}(X_1))$ and it remains to use Part a. \square

2.4. Radial maxima and minima of conjugate convex functions.

Proposition 3. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$, where $d \geq 1$, be any convex function that is finite on \mathbb{R}^d and differentiable at 0. Then Parts a and c of Proposition 2 remain valid with $F, F^*, \nabla F(0)$ substituted for K, I, μ , respectively. If in addition F is differentiable on \mathbb{R}^d , then Parts b, d, e of Proposition 2 are valid with $\bar{\Lambda}_p(F)$ and $\underline{\Lambda}_r(F^*)$ substituted for $\bar{\Lambda}_p$ and $\underline{\Lambda}_r$, and Corollary 2 holds with \mathbb{S}^{d-1} substituted for \mathbb{S}^1 .*

We wish to thank Fedor Petrov for showing us a simple proof of Part c. Originally we had two other proofs based on two very distinct geometric and analytic approaches. Both proofs were longer than the one presented here.

Proof of Proposition 3. a) $\bar{F}(p) = \max_{\ell \in \mathbb{S}^{d-1}} F(p\ell)$, $p \geq 0$, is convex as a maximum of convex functions $F_\ell(\cdot) := F(\cdot\ell)$. Further, the convex function F attains its maximum over any closed compact convex set on the boundary of the set. Therefore for any $0 \leq p < p'$, we have

$$\bar{F}(p) = \max_{u:|u|=p} F(u) = \max_{u:|u|\leq p} F(u) \leq \max_{u:|u|\leq p'} F(u) = \bar{F}(p').$$

Hence \bar{F} is increasing on $[0, \infty)$.

Note that the right derivative of \bar{F} satisfies $\bar{F}'_+(0) = |m|$, where we put $m := \nabla F(0)$.

b) For any $p > 0$ and $\ell \in \bar{\Lambda}_p(F)$, $\nabla F(p\ell)$ is directed along ℓ since $p\ell$ is an extremal point of the continuously differentiable (see Section 2.1) function F over the sphere $p\mathbb{S}^{d-1}$. Hence $|\nabla F(p\ell)| = F'_\ell(p)$ and $\bar{F}'_+(p) \geq \max_{\ell \in \bar{\Lambda}_p(F)} |\nabla F(p\ell)|$.

Further, since \mathbb{S}^{d-1} is compact, there exist two sequences $p_k \rightarrow p+$ and $\ell(k) \in \bar{\Lambda}_{p_k}(F)$ such that $\ell(k) \rightarrow \ell$ for some $\ell \in \mathbb{S}^{d-1}$ as $k \rightarrow \infty$. Then necessarily $\ell \in \bar{\Lambda}_p(F)$ since F and \bar{F} are continuous. Finally,

$$\bar{F}(p_k) - \bar{F}(p) = F(p_k \ell(k)) - F(p\ell) = (p_k \ell(k) - p\ell) \cdot (\nabla F(p\ell) + o(1)) \leq (p_k - p)(|\nabla F(p\ell)| + o(1))$$

as $k \rightarrow \infty$, and thus $\bar{F}'_+(p) \leq \max_{\ell \in \bar{\Lambda}_p(F)} |\nabla F(p\ell)|$.

The formula for $\bar{F}'_-(p)$ is analogous.

c) We first note that the convex function $\text{conv}(\underline{F}^*)$ is lower semi-continuous (of course this property should be checked only at points of $\partial \mathcal{D}_{\text{conv}(\underline{F}^*)}$). To see this, we repeat the proof of Lemma 1.b to obtain that \underline{F}^* is lower semi-continuous on \mathbb{R} and in particular, at points of $\partial \mathcal{D}_{\text{conv}(\underline{F}^*)}$. Since $\mathcal{D}_{\underline{F}^*} = \mathcal{D}_{\text{conv}(\underline{F}^*)}$, this implies the claim.

By Part a, the function \bar{F} also is convex and lower semi-continuous. Then we can apply (8), and the claim follows if we check that for any $p \geq 0$,

$$(\underline{F}^*)^*(p) = \bar{F}(p); \tag{11}$$

note that the Legendre–Fenchel transform maps $r \in [|m|, \infty)$ to $p \in [0, \infty)$ since \bar{F} is convex and increasing on \mathbb{R}_+ and $\bar{F}'_+(0) = |m|$.

Recalling that by the definition, $\underline{F}^*(r) = \infty$ for $r < 0$, we have

$$\begin{aligned} (\underline{F}^*)^*(p) &= \sup_{r \geq 0} (pr - \underline{F}^*(r)) = \sup_{r \geq 0} (pr - \inf_{\ell \in \mathbb{S}^{d-1}} F^*(r\ell)) \\ &= \sup_{r \geq 0, \ell \in \mathbb{S}^{d-1}} (pr - F^*(r\ell)) = \sup_{v \in \mathbb{R}^d} (p|v| - F^*(v)). \end{aligned}$$

On the other hand, F is convex and finite on \mathbb{R}^d and therefore continuous, hence by (8) it holds $F = F^{**}$ and we have

$$\bar{F}(p) = \sup_{\ell \in \mathbb{S}^{d-1}} F(p\ell) = \sup_{\ell \in \mathbb{S}^{d-1}} \sup_{v \in \mathbb{R}^d} (p\ell \cdot v - F^*(v)) = \sup_{v \in \mathbb{R}^d} (p|v| - F^*(v)).$$

This proves (11).

d) It is easy to see that the non-negative function $\underline{F}^* - \text{conv } \underline{F}^*$, which we define to be zero on the complement set $\mathcal{D}_{\underline{F}^*}^c$, is lower semi-continuous. This is by the lower semi-continuity of \underline{F}^* explained above and the continuity of $\text{conv } \underline{F}^*$ on $\text{rint } \mathcal{D}_{\underline{F}^*}$. Hence the set

$$\{r \geq |m| : \underline{F}^*(r) = \text{conv } \underline{F}^*(r)\} = \{r \geq |m| : \underline{F}^*(r) - \text{conv } \underline{F}^*(r) \leq 0\}$$

is closed as a sub-level set of a lower semi-continuous function. Therefore if $\underline{F}^*(r) \neq \text{conv } \underline{F}^*(r)$ for an $r \in (|m|, \sup \mathcal{D}_{\underline{F}^*})$, then this non-equality also holds on an open interval (r_1, r_2) that contains r , on which $\text{conv } \underline{F}^*$ must be affine. Since $\text{conv } \underline{F}^* = (\bar{F})^*$ on $[|m|, \infty)$ by Part c, we conclude that $(\bar{F})^*$ is affine on $[r_1, r_2]$. As we explained in Section 2.1, this yields that $r \in (r_1, r_2) \not\subset \text{cl}(\text{im}(\bar{F}'))$, which is a contradiction.

e) Let us show that $\bar{\Lambda}_p(F) \subset \bar{\Lambda}_r(F^*)$. For any $\ell \in \bar{\Lambda}_p(F)$, $\nabla F(p\ell)$ is directed along ℓ , hence by Part b it holds $\nabla F(p\ell) = r\ell$. Note that $F(u) \geq F(p\ell) + r\ell \cdot (u - p\ell)$ for any $u \in \mathbb{R}^d$ since the right-hand side of this inequality defines the support hyperplane to graph of F at the point $(p\ell, F(p\ell))$. Then

$$\begin{aligned} \underline{F}^*(r) &\leq F^*(r\ell) = \sup_{u \in \mathbb{R}^d} (r\ell \cdot u - F(u)) \leq \sup_{u \in \mathbb{R}^d} (r\ell \cdot u - F(p\ell) - r\ell \cdot (u - p\ell)) \\ &= r\ell \cdot p\ell - F(p\ell) = rp - \bar{F}(p). \end{aligned} \quad (12)$$

The concave function $q \mapsto rq - \bar{F}(q)$ attains its maximum at $q = p$ since by the assumption, it holds $(\bar{F})'(p) = r$. Then by Parts c and d,

$$rp - \bar{F}(p) = (\bar{F})^*(r) = \text{conv } \underline{F}^*(r) = \underline{F}^*(r), \quad (13)$$

and since the later expression equals the first term in (12), we get $\ell \in \bar{\Lambda}_r(F^*)$.

It remains to prove the reverse inclusion $\bar{\Lambda}_r(F^*) \subset \bar{\Lambda}_p(F)$. Suppose that $\ell \in \bar{\Lambda}_r(F^*)$. Combining the Fenchel inequality with (13), we obtain

$$\bar{F}(p) \geq F(p\ell) \geq r\ell \cdot p\ell - F^*(r\ell) = rp - \underline{F}^*(r) = \bar{F}(p),$$

which implies that $\ell \in \bar{\Lambda}_p(F)$. □

Proof of Corollary 2. Since the function \bar{F} is convex on $[0, \infty)$, its left and right derivatives satisfy ([10, Theorem 24.1])

$$\bar{F}'_+(p-) = \bar{F}'_-(p) \leq \bar{F}'_+(p) = \bar{F}'_-(p+), \quad p > 0.$$

On the other hand, $\bar{F}'_-(p) \leq |\nabla F(p\ell(p))| \leq \bar{F}'_+(p)$ by Proposition 3.b. Combining these inequalities with the continuity of the gradient ∇F concludes the proof. □

Example 2. Let us present a centred distribution with non-convex \underline{I} .

Put $X_1 = (\tilde{X}, 0)$, where \tilde{X} is a random variable with the distribution belonging to the one-parametric family that satisfies $\mathbb{P}(\tilde{X} \in \{-2, 1, 3\}) = 1$, $\mathbb{E}\tilde{X} = 0$, and $\mathbb{E}\tilde{X}^3 < 0$.

The Taylor expansion

$$\mathcal{L}((u_1, u_2)) =: \tilde{\mathcal{L}}(u_1) = 1 + \frac{1}{2}\mathbb{E}\tilde{X}^2 u_1^2 + \frac{1}{6}\mathbb{E}\tilde{X}^3 u_1^3 + o(u_1^3), \quad u_1 \rightarrow 0$$

implies that $\tilde{\mathcal{L}}(-u_1) > \tilde{\mathcal{L}}(u_1)$ for all $u_1 > 0$ that are small enough. On the other hand, $\tilde{\mathcal{L}}(-u_1) < \tilde{\mathcal{L}}(u_1)$ for all large u_1 . Hence $\tilde{\mathcal{L}}(u_*) = \tilde{\mathcal{L}}(-u_*)$ for some $u_* > 0$, and it is actually possible to choose $\mathbb{P}(\tilde{X} = 3)$ such that $\tilde{\mathcal{L}}'(u_*) \neq \tilde{\mathcal{L}}'(-u_*)$. Thus $\bar{\mathcal{L}}(r) = \min(\tilde{\mathcal{L}}(-r), \tilde{\mathcal{L}}(r))$ is not differentiable at $r = u_*$ and so \underline{I} is non-convex.

2.5. The convexity of \underline{I} for certain distributions. Here we prove Part a, b, and c of Proposition 1 by showing that \underline{I} is convex for the corresponding distributions. Part d will be proved in Section 3 using different methods. We will prove the d -dimensional version of Proposition 1.

Proof of Proposition 1. a) and b) In both cases, the random vector $R := \Sigma^{-1/2}(X_1 - \mu)$ has a rotationally invariant distribution in \mathbb{R}^d . Denote by \mathcal{L}_R its Laplace transform and put $I_R := (\log \mathcal{L}_R)^*$. Since $\mathcal{L}(u) = e^{u \cdot \mu} \mathcal{L}_R(|\Sigma^{1/2}u|)$ for $u \in \mathbb{R}^d$, it holds

$$I(v) = I_R(\Sigma^{-1/2}(v - \mu)) = \underline{I}_R(|\Sigma^{-1/2}(v - \mu)|), \quad v \in \mathbb{R}^d.$$

The function \underline{I}_R is increasing and convex on \mathbb{R}_+ , hence

$$\underline{I}(r) = \underline{I}_R\left(\min_{\ell: |\ell|=1} |\Sigma^{-1/2}(r\ell - \mu)|\right), \quad r \geq 0.$$

Then denoting by $\|\cdot\|$ the operator norm (the largest singular value) of a matrix, we have $\underline{I}(r) = \underline{I}_R(r/\|\Sigma\|^{1/2})$ for the case $\mu = 0$ and $\underline{I}(r) = \underline{I}_R(|r - |\mu||/\|\Sigma\|^{1/2})$ for the case that μ is a non-zero maximal eigenvector of Σ . Thus in these cases \underline{I} is convex because so is \underline{I}_R .

Note that in Case a, every $\underline{\Lambda}_r$ coincides with the set of unit maximal eigenvectors of Σ , and in Case b, every $\underline{\Lambda}_r$ consists of the unique element $\mu/|\mu|$. It is also easy to check directly that these statements also hold true for every $\bar{\Lambda}_p$.

c) We will first give a detailed treatment of the two-dimensional case and then consider the d -dimensional version.

1. The planar case $d = 2$.

The cumulant generating function K of a Gaussian(μ, Σ) distribution is $K(u) = u^\top \mu + \frac{1}{2}u^\top \Sigma u$. The graph of this function is an elliptic paraboloid. By Corollary 1, it suffices to show that the radial maximum function $\bar{K}(p)$ is differentiable on $(0, \infty)$.

Without loss of generality we can assume that

$$K(x, y) = a(x - x_0)^2 + b(y - y_0)^2 + c,$$

where $\mu = (-x_0, -y_0)$, $c = K(-x_0, -y_0)$, and the parameters satisfy $a > b > 0$ and $x_0, y_0 \geq 0$ with $x_0 + y_0 > 0$. The cases $x_0 = y_0 = 0$, $a = b$, and $b = 0$ are trivial (the first two are already covered above by Cases a and b).

Suppose that $\ell \in \bar{\Lambda}_p$. The gradient $\nabla K(x, y) = (2a(x - x_0), 2b(y - y_0))$ of K at the point $p\ell = (x, y)$ is proportional to ℓ . Then $2a(x - x_0)y = 2b(y - y_0)x$, which is equivalent to

$$h(x, y) = (a - b)xy - ax_0y + by_0x = 0. \quad (14)$$

This equation defines the hyperbola \mathcal{H} with asymptotes parallel to the coordinate axes:

$$y = -\frac{by_0}{a - b}, \quad x = \frac{ax_0}{a - b}, \quad (15)$$

see Figure 3. It is the Apollonian hyperbola (see Glaeser et al. [7, section 9.3] for details) for the ellipses that are contour lines of K .

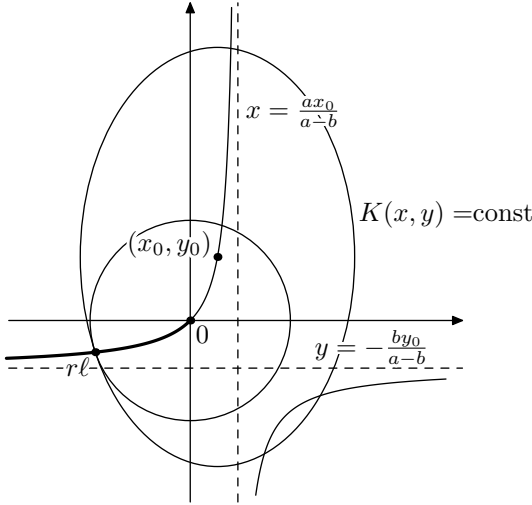


Fig. 3.

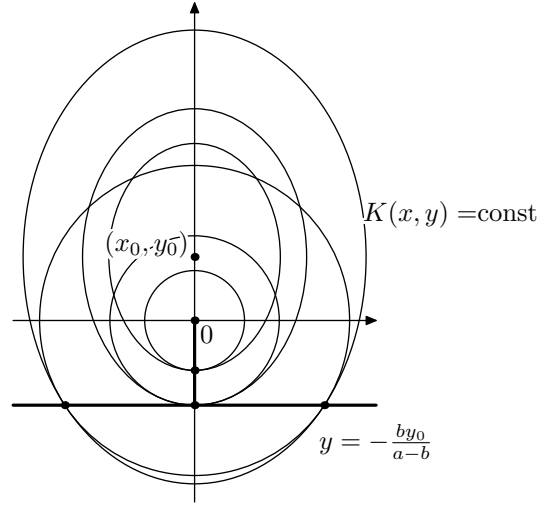


Fig. 4.

Consider the case $x_0, y_0 > 0$.

The set $\bar{\Lambda}_p$ lies in the quadrant $\{(x, y) : x \leq 0, y \leq 0\}$ since $K(-|x|, -|y|) < K(x, y)$ for any pair (x, y) in the complement of the quadrant. Because every $\bar{\Lambda}_p$ is non-empty, from the necessary condition (14) we conclude that the set $\cup_{p>0} p\bar{\Lambda}_p$ is the arc of \mathcal{H} that belongs to the interior of the quadrant (marked in bold in Figure 3), and each set $\bar{\Lambda}_p$ contains exactly one direction. Since this direction varies continuously in p , the function \bar{K} is differentiable by Corollary 2.

In the cases $x_0 > 0, y_0 = 0$ and $x_0 = 0, y_0 > 0$, the hyperbola \mathcal{H} degenerates into two lines (15). Below we will present a different method which covers such generate cases in any dimension so the following argument is given for completeness of consideration.

It is easy to see that in the first case, $\bar{\Lambda}_p = \{(-1, 0)\}$ for every $p > 0$; this situation is actually covered above by Case b since $a > b$.

In the second case, both solutions $x = 0$ and $y = -\frac{by_0}{a-b}$ contribute to the answer: we have $\bar{\Lambda}_p = \{(0, -1)\}$ for $p \in (0, by_0/(a - b)]$ and for $p > by_0/(a - b)$, the sets $\bar{\Lambda}_p$ consist of two directions symmetric about the y -axis with non-zero x -coordinates. The set $\cup_{p>0} p\bar{\Lambda}_p$ is marked in bold in Figure 4.

Clearly, in both cases there is a continuous path of directions $\ell(p) \in \bar{\Lambda}_p$ so \bar{K} is differentiable by Corollary 2.

2. The general case $d \geq 3$.

The case of degenerate Σ is by reduction of the dimension.

For non-degenerate Σ and positive coordinates of $-\mu$ in the basis of eigenvectors of Σ , the argument follows the same scheme as above. We obtain $d - 1$ of equations of type (14) for each pair of coordinates x_1 and x_i , $2 \leq i \leq d$. The set of solution of these equations with negative coordinates is a simple curve that coincides with $\cup_{p>0} p\bar{\Lambda}_p$. Each coordinate x_i , $2 \leq i \leq d$, of a point on this curve is monotone in x_1 ; in particular, if the largest eigenvalue of Σ has multiplicity $k \geq 2$, then the coordinates x_i , $2 \leq i \leq k$, change linearly. Hence the distance p from the point to the origin is also monotone in x_1 . So every $\bar{\Lambda}_p$ contains exactly one direction and we can continuously parametrize this direction by p .

For the remaining case that Σ is non-degenerate and some coordinates of μ are zero, we proceed differently from the previous consideration and prove the convexity of \underline{I} directly rather than using Corollary 2.

The rate function of a Gaussian(μ, Σ) distribution is given by $I(v) = \frac{1}{2}(v - \mu)^\top \Sigma^{-1}(v - \mu)$. For any $\varepsilon > 0$, the rate function $I_\varepsilon(v) := I(v + \varepsilon e_d)$, where $e_d := (1, \dots, 1)$, corresponds to $\mu_\varepsilon := \mu - \varepsilon e_d$ with negative coordinates, hence $\underline{I}_\varepsilon$ is convex on $[0, \infty)$. For any $R > 0$, denote by B_R the ball of radius R around the origin. Since $I_\varepsilon \rightarrow I$ as $\varepsilon \rightarrow 0+$ uniformly on every ball B_R , it clearly holds $\underline{I}_\varepsilon \rightarrow \underline{I}$ uniformly on every $[0, R]$. Then \underline{I} is continuous and convex on $[0, R]$, and hence it is convex on $[0, \infty)$.

d) This case will be considered in Section 3 using different methods.

□

3. PROOFS OF THE MAIN RESULTS

Let us recall the functional large deviations principle for trajectories of random walks. Denote by $C[0, 1] = C([0, 1], \mathbb{R}^2)$ the space of continuous functions from $[0, 1]$ to \mathbb{R}^2 , i.e. planar curves, equipped with the usual metric of uniform convergence. Let $S_n(\cdot) \in C[0, 1]$ be the random piecewise linear functions that satisfy $S_n(k/n) := S_k$, $0 \leq k \leq n$, and their values at points $t \notin \{0, 1/n, \dots, 1\}$ are defined by linear interpolation. Define the rate function $I_C : C[0, 1] \rightarrow [0, \infty]$ to be

$$I_C(h) := \begin{cases} \int_0^1 I(h'(t)) dt, & \text{if } h \text{ is absolutely continuous and } h(0) = 0; \\ +\infty, & \text{if otherwise.} \end{cases}$$

This is a lower semi-continuous function with compact sub-level sets $\{h : I_C(h) \leq \alpha\}$, $\alpha \in \mathbb{R}$, see Dembo and Zeitouni [5, Theorems 4.2.1 and 5.1.2]. Denote by $C_0[0, 1]$ the subspace of continuous functions h on $[0, 1]$ such that $h(0) = 0$, and by $AC_0[0, 1]$ the subspace of $C_0[0, 1]$ of absolutely continuous functions.

The functional large deviations principle for random walks states that (combine Theorem 5.1.2 with Theorem 4.2.13 and Lemma 5.1.7 of [5]) for any Borel set $B \subset C[0, 1]$,

$$\begin{aligned} - \inf_{h \in \text{int}(B)} I_C(h) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n(\cdot)/n \in B) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n(\cdot)/n \in B) \leq - \inf_{h \in \text{cl}(B)} I_C(h). \end{aligned} \quad (16)$$

In particular, if B is *regular* for the rate function I_C , that is

$$\inf_{h \in \text{int}(B)} I_C(h) = \inf_{h \in \text{cl}(B)} I_C(h), \quad (17)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n(\cdot)/n \in B) = - \inf_{h \in B} I_C(h). \quad (18)$$

This result is due to Mogulskii [8].

This functional large deviations principle for the trajectories $S_n(\cdot)$ in $C[0, 1]$ readily implies the large deviations principle for random vectors S_n in \mathbb{R}^2 . Similarly to (17), a Borel set $A \subset \mathbb{R}^2$ is regular for the rate function I if $\inf_{v \in \text{int}(A)} I(v) = \inf_{v \in \text{cl}(A)} I(v)$. By the Jensen inequality, for any absolutely continuous functions $h \in AC_0[0, 1]$, we have

$$I_C(h) = \int_0^1 I(h'(t)) dt \geq I(h(1)), \quad (19)$$

and thus (18) implies that for any regular set $A \subset \mathbb{R}^2$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n/n \in A) = - \inf_{v \in A} I(v). \quad (20)$$

Moreover, it is known that (20) holds for any convex open set A .

In particular, the later fact implies the statement of Proposition 1.d on *one-dimensional* random walks. Indeed, in this case we may assume without loss of generality that $S_n =: (\tilde{S}_n, 0)$, where \tilde{S}_n is a random walk in \mathbb{R} . Then it is easy to see that for any $x \geq |\mu|$, we have

$$\mathbb{P}(|\tilde{S}_n| \geq 2xn) \leq \mathbb{P}(P_n \geq 2xn) \leq n\mathbb{P}(|\tilde{S}_n| \geq 2xn),$$

and Proposition 1 immediately follows if we apply (20) for the half-planes $\pm[x, \infty) \times \mathbb{R}$ and use that $\underline{I}(x) = \min(I(x, 0), I(-x, 0))$, which holds by $\mathcal{D}_I \subset \mathbb{R} \times \{0\}$.

Proof of Theorem 1. Let $P(h)$ denote perimeter of the convex hull of the image of a function $h \in C[0, 1]$. Our first goal is to find the infima of I_C over the sets $\{h : P(h) \leq 2x\}$ and $\{h : P(h) \geq 2x\}$.

1. Let us fix any $x \in (r_{\min}, |\mu|]$.

For any function $h \in C_0[0, 1]$, it clearly holds $P(h) \geq 2|h(1)|$. Then by (19) and the fact that \underline{I} is strictly decreasing on $[0, |\mu|]$,

$$\inf_{h: P(h) \leq 2x} I_C(h) \geq \inf_{h: P(h) \leq 2x} I(h(1)) \geq \inf_{h: |h(1)| \leq x} I(h(1)) = \inf_{r \leq x} \underline{I}(r) = \underline{I}(x).$$

These inequalities are actually equations: the Jensen inequality (19) is an equation for linear h , and for any $r \geq 0$ and $\ell \in \underline{\Lambda}_r$, we have

$$I_C(r\ell t) = \underline{I}(r), \quad P(r\ell t) = 2r. \quad (21)$$

Hence it holds

$$\inf_{h: P(h) \leq 2x} I_C(h) = \underline{I}(x), \quad x \leq |\mu|. \quad (22)$$

Moreover, we have

$$(I_C(h) = \underline{I}(x), P(h) \leq 2x) \quad \text{iff} \quad h = x\ell_x t. \quad (23)$$

Indeed, since \underline{I} is strictly decreasing on $(r_{\min}, |\mu|]$, the infimum in (22) can only be attained on functions $h \in C_0[0, 1]$ that satisfy $P(h) = 2|h(1)| = 2x$, that is $h(t) = x\ell t$ for some $\ell \in \mathbb{S}^1$. The unique h that satisfies the first equation in (23) corresponds to the direction ℓ_x .

2. Let us fix any $x \in [|\mu|, r_{\max})$.

Our main estimate follows from the inequality $I(v) \geq \text{conv } \underline{I}(|v|)$, which holds for every $v \in \mathbb{R}^2$, and the Jensen inequality applied to the convex function $\text{conv } \underline{I}$: for any $h \in AC_0[0, 1]$, we have

$$I_C(h) = \int_0^1 I(h'(t)) dt \geq \int_0^1 \text{conv } \underline{I}(|h'(t)|) dt \geq \text{conv } \underline{I}\left(\int_0^1 |h'(t)| dt\right) = \text{conv } \underline{I}(V(h)), \quad (24)$$

where V denotes total variation of h on $[0, 1]$ (which by definition equals length of the curve h). Note that $V(h)$ does not depend on the parametrization of the curve but $I_C(h)$ does.

It remains to use the following well-known inequality, which is even referred to as geometric “folklore”: it holds $V(h) \geq \frac{1}{2}P(h)$ for any $h \in C[0, 1]$ of bounded variation, see Corollary 3 of Appendix. Since the function \underline{I} strictly increases on $[|\mu|, r_{\max})$ so does its largest convex minorant $\text{conv } \underline{I}$. These two facts imply that

$$\inf_{h: P(h) \geq 2x} \text{conv } \underline{I}(V(h)) \geq \text{conv } \underline{I}(x). \quad (25)$$

Finally, combining (24) with (25) and using (21) for the upper bound, we get

$$\text{conv } \underline{I}(x) \leq \inf_{h: P(h) \geq 2x} I_C(h) \leq \underline{I}(x), \quad x \geq |\mu|. \quad (26)$$

We claim that if $\underline{I}(x) = \text{conv } \underline{I}(x)$, then

$$(I_C(h) = \underline{I}(x), P(h) \geq 2x) \quad \text{iff} \quad h \in \{x\ell t, \ell \in \underline{\Lambda}_x\}. \quad (27)$$

To prove this statement, we first note that since $\text{conv } \underline{I}$ is strictly increasing on $[|\mu|, r_{\max})$, the equality in (25) is attained only on the functions $h \in C_0[0, 1]$ that satisfy $V(h) = \frac{1}{2}P(h) = x$. By Corollary 3 of Appendix, such functions have the form $h(t) = |h(t)|\ell$ a.e. t for some $\ell \in \mathbb{S}^1$ and satisfy $V(h) = x$. Further, the second inequality in (24) is an equation iff $|h'(t)| \in [x_1, x_2]$ a.e. t , where $[x_1, x_2]$ is the maximal by inclusion interval that contains x and is such that the restriction of $\text{conv } \underline{I}$ on $[x_1, x_2]$ is affine. Finally, the first inequality in (24) is an equation for a function $h \in C_0[0, 1]$ that satisfies the conditions above iff

$$|h'(t)| \in L_x := \{y \in [x_1, x_2] : I(y\ell) = \text{conv } \underline{I}(y)\} \quad \text{a.e. } t$$

with the direction ℓ which was already fixed above. Since the rate function I is strictly convex, so is $I(\cdot\ell)$, hence $L_x = \{x\}$. Thus we obtained that $|h'(t)| = x$ a.e. t and by $I(x\ell) = \underline{I}(x)$, we have $\ell \in \underline{\Lambda}_x$. This finishes the proof of (27).

We are now ready to prove (2) and (4). Note that by

$$\text{conv}(\{S_n(t)\}_{0 \leq t \leq 1}) = \text{conv}(S_0, S_1, \dots, S_n),$$

it holds that

$$P(S_n(\cdot)/n) = P_n/n. \quad (28)$$

It readily follows by the Cauchy formula (30) of Appendix for perimeter of a planar convex shape that the functional P is continuous on $C[0, 1]$. Hence the sets $\{h : P(h) \geq 2x\}$ and $\{h : P(h) \leq 2x\}$ are closed in $C[0, 1]$ and they share the same boundary $\{h : P(h) = 2x\}$.

It follows from (22) that the sets $\{h : P(h) \leq 2x\}$ are regular for any $x \in (r_{\min}, |\mu|]$ by the continuity of \underline{I} on this interval, which is due to the fact that \underline{I} is convex on this interval, see Lemma 1.b. Then (2) follows by (18) and (28).

In general, we can not assure the regularity of $\{h : P(h) \geq 2x\}$ for an $x \in [|\mu|, r_{\max})$. The upper bound in (4) immediately follows by the lower bound in (26) and the upper bound in the general large deviations principle (16). For the lower bound in (4), we need to consider two cases. If \underline{I} is continuous at x , then we use the upper bound in (26) and the lower bound in (16). If \underline{I} is discontinuous at x , then by Lemma 2.b, the distribution of X_1 has atoms at the points of $x\underline{\Lambda}_x$ and $\underline{I}(x) = -\log \mathbb{P}(X_1 = x\ell)$ for any $\ell \in \underline{\Lambda}_x$. Then we have

$$\mathbb{P}(P_n \geq 2xn) \geq \mathbb{P}(S_k = kx\ell, k = 1, \dots, n \text{ for some } \ell \in \underline{\Lambda}_x) = \#(\underline{\Lambda}_x)e^{-n\underline{I}(x)},$$

which gives the lower bound in (4). This argument of course works for $x = r_{\max}$ proving the statements of Remark 1. The proof of (4) and (2) is now finished.

It remains to prove (3) and (5). By the continuity of the distance function $\inf_{\ell \in \underline{\Lambda}_x} \|x\ell t - \cdot\|_C$ on $C[0, 1]$, the sets

$$B_x^{\geq \varepsilon} := \{h : \inf_{\ell \in \underline{\Lambda}_x} \|h - x\ell t\|_C \geq \varepsilon\}, \quad B_x^{< \varepsilon} := \{h : \inf_{\ell \in \underline{\Lambda}_x} \|h - x\ell t\|_C \leq \varepsilon\}$$

are closed and have the same boundary $\{h : \inf_{\ell \in \underline{\Lambda}_x} \|h - x\ell t\|_C = \varepsilon\}$. Then by (16), for $x \geq [|\mu|, r_{\max})$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}\left(\max_{1 \leq k \leq n} \left|\frac{S_k}{n} - \frac{k}{n}x\ell\right| > \varepsilon \text{ for any } \ell \in \underline{\Lambda}_x, P_n \geq 2xn\right) \leq - \inf_{h \in B_x^{\geq \varepsilon} : P(h) \geq 2x} I_C(h), \quad (29)$$

and (5) follows if we show that the infimum is strictly larger than $\underline{I}(x) < \infty$.

Assuming the converse, we find that $\underline{I}(x)$ equals the infimum of I_C over the set $B_x^{\geq \varepsilon} \cap \{h : P(h) \geq 2x, I_C(h) \leq \underline{I}(x) + 1\}$, which is compact as a closed subset of the compact sub-level set $\{h : I_C(h) \leq \underline{I}(x) + 1\}$. On this set, the lower semi-continuous function I_C attains its minimal value, hence there exists an $h_0 \in C_0[0, 1]$ such that $I_C(h_0) = \underline{I}(x)$ and $P(h_0) \geq 2x$ but $h_0 \notin \{x\ell t, \ell \in \underline{\Lambda}_x\}$. This is a contradiction with (27).

The proof of (3) is completely analogous: simply replace all the three inequalities “ \geq ” in (29) by “ \leq ” and use (23) instead of (27). \square

The proof of Remark 3 goes exactly as the above, with the only difference that Remark 4 of Appendix should be used instead of Corollary 3.

It is unclear if there is any possibility for improvement in the presented approach: it is possible to construct a smooth convex function F on \mathbb{R}^2 such that

$$\min_{0 \leq s \leq 1} \left[sF\left(\frac{u_1}{s}\right) + (1-s)F\left(\frac{u_2}{1-s}\right) \right] \not\geq F\left(\frac{1}{2}(|u_1| + |u_2| + |u_1 + u_2|)\right), \quad u_1, u_2 \in \mathbb{R}^2.$$

Hence in order to strengthen (24), some finer properties of the rate function I must be used besides its convexity.

Proof of Theorem 2. Since by our assumption, \underline{I} is convex, (24) holds true with \underline{I} substituted for $\text{conv } \underline{I}$. Denote by $A(h)$ the area of the convex hull of the image of a function $h \in C[0, 1]$. Any function $h \in C[0, 1]$ of bounded variation satisfies the isoperimetric inequality $A(h) \leq V(h)^2/(2\pi)$, where the equation is attained iff h parametrizes a semi-circle: this is Ulam's version of the classical Dido problem solved by Moran [9]; see Tilli [14] for the "iff" statement. It remains to use that A is a continuous functional on $C[0, 1]$. The claim then follows since the second inequality in (24) is an equation iff $|h'(t)|$ is constant for a.e. t . \square

Example 3. Let us discuss the limit shape of trajectories that result in large deviations of the area A_n under the assumption that X_1 has a shifted standard Gaussian distribution with $\mu \neq 0$. In this case $I(v) = \frac{1}{2}|v - \mu|^2$, and we assume without loss of generality that $\mu = (|\mu|, 0)$.

We need to minimize $I_C(h)$ over $\{h \in AC_0[0, 1] : A(h) \geq a\}$. Suppose that the minimum is attained at an $h(t) = (x(t), y(t))$ and suppose that $h(t)$ is twice continuously differentiable. By the axial symmetry, it holds $y(t) = y(1-t)$ on $[0, 1]$ so $y(1) = 0$.

Assume that h is a convex curve, which means there are no three points lying on a line. In particular, this assumption implies that the curve lies to one side of the x -axis. It is by far not obvious that a minimizer must satisfy this assumption – it took a certain effort, which is the essence of Moran's paper [9], to verify it in the simpler case of Ulam's problem.

Then the curve formed by h and the part of the horizontal axis between 0 and $h(1)$ is a simple closed curve which bounds the region of area $A(h) = \frac{1}{2} \left| \int_0^1 (xy' - x'y) dt \right|$.

A necessary condition that such h minimizes $I_C(h)$ on $\{h \in AC_0[0, 1] : A(h) = a\}$ is that it satisfies the Euler–Lagrange equations with the boundary condition $y(1) = 0$ for the Lagrange function

$$H(x, x', y, y') := (x' - |\mu|)^2 + (y')^2 + \frac{\lambda}{2}(xy' - x'y)$$

with some $\lambda \in \mathbb{R}$. It is not hard to show that the solutions are two circular arcs parametrized with a constant speed (like for the classical Dido problem with fixed end points) but the radius and the angle of these arcs are not explicit in terms of $|\mu|$ and a . Note that the quantity $|h'(t) - \mu|$ is not constant for such motions unlike the minimizers appearing in the proofs of Theorems 1 and 2.

It is very plausible that the circular arcs indeed minimize $I_C(h)$ for a fixed area of the convex hull. However, one still needs to give a rigorous proof of existence of the minimizer

and its smoothness and, importantly, overcome the convexity assumption we made above. We leave these questions aside.

APPENDIX

The following simple proposition is proved in our separate note [1], which was initially motivated by the problem on convex hulls of random walks considered in this paper. For the reader's convenience, we present the result here. Its use for the current paper is in the corollary, which not only gives the “folklore” inequality for the half-perimeter but also specifies all instances when the equation is attained.

Proposition 4. *Let γ be a rectifiable curve in \mathbb{R}^2 , and let Γ denote its convex hull. Then*

$$\text{length } \gamma \geq \text{per } \Gamma - \text{diam } \Gamma.$$

Corollary 3. *It holds*

$$\text{length } \gamma \geq \frac{1}{2} \text{per } \Gamma,$$

and equation can be attained only if γ parametrizes is a line segment.

Remark 4. These statements remain valid if we replace \mathbb{R}^2 by \mathbb{R}^d (with any $d \geq 2$) and $\text{per } \Gamma$ by $\frac{dv_d}{v_{d-1}} W(\Gamma)$, where v_d denotes volume of a unit ball in \mathbb{R}^d and

$$W(\Gamma) := \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} w_\ell(\Gamma) d\ell$$

is mean width of Γ , with $w_\ell(\Gamma)$ being width of Γ in the direction ℓ i.e. length of the projection of Γ on the line passing through the origin in the direction ℓ . The normalizing factor corresponds to mean width $\frac{2v_{d-1}}{dv_d}$ of a unit segment in \mathbb{R}^d .

The proof easily follows by the extension to higher dimensions of the Crofton formula (which can be verified from Santaló [11, Eq. (13.9) and (13.49)]):

$$\text{length } \gamma = \frac{1}{v_{d-1}} \iint_{\mathbb{S}^{d-1} \mathbb{R}_+} n_\gamma(\ell, r) d\ell dr,$$

where $n_\gamma(\ell, r)$ denotes the number of intersections of γ with the hyperplane perpendicular to the direction ℓ at the distance r from the origin. Joining the end points of γ by a line segment turns it into a closed curve γ' , and then almost every hyperplane that intersects Γ does intersect γ' at least at two points since $\text{conv}(\gamma') = \Gamma$. It remains to use that $|\mathbb{S}^{d-1}| = dv_d$.

Note that the Crofton formula implies the Cauchy formula for perimeter of a planar convex shape Γ :

$$\text{per } \Gamma = \frac{1}{2} \int_{\mathbb{S}^1} w_\ell(\Gamma) d\ell. \tag{30}$$

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